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A New Method of Analysis of the Effect of Weak Colored Noise in Nonlinear Dynamical Systems

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A systematic method for obtaining the asymptotic behavior of a dynamical system forced by colored noise in the limit of small intensity is developed. It is based on the search of WKB solutions to the Fokker–Planck equation for the joint probability density of the system and noise, in which the perturbation expansion is continued to the first correction beyond the Hamilton–Jacobi limit. The method can be applied to noise with correlation time of order unity. It is illustrated on the normal form of a pitchfork bifurcation, where it is pointed out that additive noise can induce a shift of the most probable value. This prediction is confirmed by numerical simulation of the stochastic differential equations.

KEY WORDS: Stochastic processes; colored noise; nonlinear dynamics.

1. INTRODUCTION

The study of dynamical systems subjected to colored noise is attracting growing interest.[1,2] While the internal fluctuations generated spontaneously by a physical system can be described adequately in terms of a white noise source,[3-5] there is in general no reason that environmental disturbances impinging on a system should share similar properties. For instance, turbulence, one of the universal causes of environmental fluctuations, has a complex spectrum in both the frequency and wave number domain.[6]

The quantitative theory of nonlinear dynamical systems involving one variable and subjected to white noise is a highly developed subject, due to the possibility of appealing to the theory of Markovian processes.[2] The situation is less satisfactory for colored noise, in which only the pair formed by the state variable and the noise is Markovian. A number of interesting

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results on this problem are available in the literature. However, these results are limited either to the vicinity of white noise[7–13] or to the opposite limit of very large correlation times.[2,7,14] The difficulty is to find a closed evolution equation for the probability density of the state variable alone. Usually, the vicinity of the white noise has to be assumed in order to ensure closure.[8–11] It is true that starting from a modified Liouville equation it is possible to obtain a closed, non-Markovian evolution equation for the probability density of the state variable[12] valid for finite correlation times; again, however, the smallness of the correlation time has to be assumed in order to achieve further reduction to a Markovian equation.[13] It is the purpose of the present paper to introduce a systematic method for tackling colored noise in the limit of small intensity. This new method is applicable in a wide range of problems and allows one to analyze the local effects around a given attractor of perturbations having any correlation time.

The method is presented in Section 2. It is based on previous work on the solution of the master equation for internal fluctuations.[15,16,20,21] Although, as stressed earlier and as will become clear in the sequel, its applicability is very wide, attention is focused in Section 2 on dynamical systems involving a single variable operating in the vicinity of a simple bifurcation point. The local solution of the bivariate Fokker–Planck equation describing the joint evolution of the system and the noise is derived in a limit involving an appropriate combination of the variance and the correlation time. From this solution, the shift of the most probable value from its deterministic limit is deduced.

In Section 3 the general results are confronted with the behavior of exactly solvable models. Full agreement is found to exist. In Section 4 we consider the particularly important case of the normal form of a pitchfork bifurcation. The surprising result that additive colored noise can induce a shift in the most probable value is confronted successfully with the results of numerical simulation. A brief discussion is presented in Section 5.

2. GENERAL APPROACH

We consider a nonlinear system involving one variable x and subjected to external noise. The evolution of such a system is described in terms of the stochastic differential equation

$$dx/dt = f(x) + g(x)\,z(t) \tag{1}$$

where $f(x)$ is a nonlinear function of x and $z(t)$ is a random process modeling the fluctuating environment. If $g(x)$ is constant, we have additive noise, while if $g(x)$ depends nontrivially on x, we have multiplicative noise (typically this is the case when the control parameters of the system are fluctuating quantities). The evolution of the system is Markovian if and only if the random process $z(t)$ is a white noise[2] [in this case Eq. (1) is often called a Langevin equation]. White noise is only an idealization of a more realistic noise with nonvanishing correlation time, that is, colored noise. In the present paper, we investigate the effects of colored noise on the stationary properties of the system. We assume that though nonwhite, the environment is still Markovian, so that we can write a Langevin equation for the z-process. The specific Markovian process we consider is a stationary, Gaussian process. Following Doob's theorem,[2] $z(t)$ is then necessarily an Ornstein–Uhlenbeck (OU) process satisfying

$$dz/dt = -\gamma z + F(t) \tag{2}$$

where $F(t)$ is a Gaussian white noise whose correlation function is given by

$$\langle F(t)\, F(t')\rangle = \varepsilon\gamma^2\delta(t-t') \tag{3}$$

The correlation function of the OU process is then

$$\langle z(t)\, z(t+\tau)\rangle = \tfrac{1}{2}\varepsilon\gamma e^{-\gamma\tau} \tag{4}$$

The main difficulty when dealing with colored noise is that the evolution of the variable x is no longer Markovian, so that there is no general method to study the behavior of a system perturbed by colored noise. On the other hand, it is always possible to transform a non-Markovian process into a Markovian one containing more variables. In particular, from Eqs. (1) and (2), the two-variable (x, z) process is Markovian. Its probability density $P(x, z, t)$ therefore satisfies the Fokker–Planck equation:

$$\frac{\partial}{\partial t} P(x, z, t) = -\frac{\partial}{\partial x}[f(x) + g(x)z]P + \gamma\frac{\partial}{\partial z} zP + \frac{1}{2}\varepsilon\gamma^2\frac{\partial^2}{\partial z^2} P \tag{5}$$

For arbitrary functions $f(x)$ and $g(x)$, the solution of such a Fokker–Planck equation is not known in general. Thus we have to resort to perturbative methods for obtaining approximate solutions to Eq. (5). We follow here the Hamilton–Jacobi method proposed by Kubo *et al.*[15] and developed further by Lemarchand and Nicolis,[20,21] Graham and Schenzle,[17] and Turner.[23] The starting point is to consider ε as a smallness parameter and seek for WKB-like solutions in the form[16-18]

$$P(x, z, t) = Ne^{-(1/\varepsilon)U(x,z,t)} \tag{6}$$

where N is a normalization constant and $U(x, z, t)$ will subsequently be referred to as the *stochastic potential.* From the standpoint of the noise process, ε is the variance of the colored noise in the limit $\gamma \to \infty$ (white noise limit). This means that in the limit of vanishing correlation time, the system would be subjected to external white noise with variance ε. On the other hand, we are in the position to investigate colored noise with arbitrary correlation time as long as the product $\varepsilon\gamma^2$ remains small.

We now expand the stochastic potential U in the form

$$U(x, z, t) = U_0(x, z, t) + \varepsilon U_1(x, z, t) \tag{7}$$

Inserting expressions (6) and (7) into Eq. (5) and identifying equal powers of ε yields

$$-\frac{\partial U_0}{\partial t} = [f(x) + g(x)z]\frac{\partial U_0}{\partial x} - \gamma z\frac{\partial U_0}{\partial z} + \frac{1}{2}\gamma^2\left(\frac{\partial U_0}{\partial z}\right)^2 \tag{8a}$$

$$-\frac{\partial U_1}{\partial t} = (f + gz)\frac{\partial U_1}{\partial x} - \gamma z\frac{\partial U_1}{\partial z} + \gamma^2\frac{\partial U_0}{\partial z}\frac{\partial U_1}{\partial z} - \frac{1}{2}\gamma^2\frac{\partial^2 U_0}{\partial z^2} - (f' - \gamma + g'z) \tag{8b}$$

The zeroth order in ε, Eq. (8a) is an equation of the Hamilton–Jacobi type. In general, the solutions of such an equation are unknown. Moreover, U_0 is not globally differentiable in general.[19] In the following we will bypass these difficulties by dealing with a local theory around one steady state. To proceed further, we assume therefore that the "deterministic" system

$$dx/dt = f(x) \tag{9}$$

possesses a stable stationary solution $\bar{x}$ around which $f(x)$ and $g(x)$ are analytical functions [note that generally $f(x)$ and $g(x)$ are polynomials, so that the assumption of analyticity is rather mild]. Expanding $f(x)$ and $g(x)$ around this point and keeping the first nontrivial contributions, we obtain

$$\begin{aligned} f(x) &= f'(x - \bar{x}) + \tfrac{1}{2}f''(x - \bar{x})^2 + \cdots \\ g(x) &= g + g'(x - \bar{x}) + \cdots \end{aligned} \tag{10}$$

where

$$f' = \left.\frac{df}{dx}\right|_{\bar{x}}, \qquad g = g(\bar{x}), \quad \text{etc.}$$

A typical situation arises when the dynamical system admits a bifurcation[4]: for a critical value of some control parameter a stationary state assumed here to be the trivial solution $x=0$ loses its stability, whereas new nontrivial stationary states emerge. A characteristic feature of our analysis is that the expansions in Eqs. (10) are carried out around one of these new stable stationary states. This will allow us to obtain a probability density that is locally normalizable and to avoid logarithmic singularities in the expansion of the stochastic potential U.

Hereafter we are interested primarily in the stationary properties of the system (1). The extrema of the stationary probability density are valuable indicators of the effect of noise on the steady state of the system.[2] Our objective is thus to obtain a general approximate expression for the shift of the most probable value of the x-process up to first order in ε. As it turns out, if the analysis is limited to U_0, the shift always vanishes. We must therefore include contributions coming from U_1. Now, as seen from Eq. (8b), to compute U_1 to the dominant order, contributions to U_0 beyond the quadratic terms are needed. Hence, in calculating U_0 we have to extend the Taylor expansion around x to such higher order terms. In fact, in order to ensure normalization, the expansion should be pursued until quartic terms. The evaluation of these terms is in general extremely arduous. Fortunately, up to first order in ε they do not affect the most probable value, so that an explicit expression of quartic terms is not necessary for our purpose. Denoting $X=x-\bar{x}$, we therefore expand U_0 in the form:

$$U_0(X,z)=\tfrac{1}{2}a_1X^2+a_2Xz+\tfrac{1}{2}a_3z^2+\tfrac{1}{6}b_1X^3+\tfrac{1}{2}b_2X^2z + \tfrac{1}{2}b_3Xz^2+\tfrac{1}{6}b_4z^3+\text{quartic terms} \tag{11}$$

The absence of first-order terms in this expression is related to the fact that U_0 is extremum on the deterministic stationary states, as can be seen directly from Eq. (8a). Inserting Eqs. (10) and (11) into Eq. (8a), we obtain a set of equations for the seven coefficients appearing in Eq. (11). The procedure for solving these equations follows straightforwardly the method developed by Lemarchand and Nicolis. After some algebra, we obtain

$$\begin{aligned}U_0(X,z)= &-\frac{f'}{\gamma^2g^2}(f'-\gamma)^2X^2-2\frac{f'}{\gamma^2g}(f'-\gamma)Xz-\frac{1}{\gamma^2}(f'-\gamma)z^2\\ &+\frac{1}{3}\frac{1}{\gamma^2g^3}\frac{(f'-\gamma)^2}{(2f'-\gamma)(f'-2\gamma)}\\ &\times(20f'^3g'-28\gamma f'^2g'-14f'^2f''g+13\gamma f'f''g+8\gamma^2f'g'-2\gamma^2f''g)X^3\end{aligned}$$

$$+\frac{1}{\gamma^2 g^2}\frac{(f'-\gamma)}{(2f'-\gamma)(f'-2\gamma)}$$

$$\times\,(10f'^3g'-14\gamma f'^2g'-8f'^2f''g+9\gamma f'f''g+4\gamma^2f'g'-2\gamma^2f''g)\,X^2z$$

$$+\frac{2}{\gamma^2 g}\frac{(f'-\gamma)}{(f'-2\gamma)}(f'g'-f''g)\,Xz^2$$

$$+\frac{2}{3}\frac{1}{\gamma^2}\frac{(f'-\gamma)}{(2f'-\gamma)(f'-2\gamma)}(f'g'-f''g)\,z^3+\text{quartic terms} \tag{12}$$

In an analogous manner, we evaluate U_1 from Eq. (8b),

$$U_1(X,z)=\frac{1}{g}\frac{1}{(2f'-\gamma)(f'-2\gamma)}(6f'^2g'-4f'f''g-8\gamma f'g'$$

$$+\,3\gamma f''g+2\gamma^2g')X+\frac{1}{(2f'-\gamma)(f'-2\gamma)}(f'g'-f''g)z$$

$$+\text{ quadratic terms} \tag{13}$$

Note that, as for the quartic terms in U_0, quadratic terms in U_1 contribute to the shift of the most probable value only to higher orders in ε. We thus have a local expression for the bivariate stochastic potential.

As mentioned above, our aim is to estimate the perturbation of the state variable X due to colored noise. We therefore construct from the bivariate probability density the probability density of a single variable X and z:

$$P(X)=\int_{-\infty}^{\infty}P(X,z)\,dz \tag{14}$$

$$P(z)=\int_{-\infty}^{\infty}P(X,z)\,dX \tag{15}$$

with $P(X,z)$ given by Eqs. (6), (12), and (13). Note that in Eq. (15) we have extended the domain of integration to the entire real axis, disregarding the boundaries of the x-process. Assuming inaccessible boundaries, this procedure is justified because $P(X,z)$ decays exponentially around its extrema, so that, for ε small enough, the modification introduced is smaller than any power of ε.

A necessary condition for the expansion (11) to be consistent is that Eq. (15) yields the same expression of $P(z)$ as the exact one deduced from Eqs. (2) and (3), namely

$$P(z)=(\pi\varepsilon\gamma)^{-1/2}\exp[-(1/\varepsilon\gamma)\,z^2] \tag{16}$$

To check this fact, we estimate asymptotically expression (15). For fixed z, we compute the extremum of $U(X, z)$:

$$\bar{X} = -\frac{g}{(f'-\gamma)} z - \frac{1}{2} g \frac{1}{(2f'-\gamma)(f'-2\gamma)(f'-\gamma)^2} \times (-4f'^2 + 2f'f''g + 6\gamma f'g' - \gamma f''g - 2\gamma^2 g') z^2 + \frac{1}{2}\gamma^2 g \frac{\varepsilon}{(2f'-\gamma)(f'-2\gamma)(f'-\gamma)^2} \times (6f'^2 g' - 4f'f''g - 8\gamma f'g' + 3\gamma f''g + 2\gamma^2 g') \tag{17}$$

In a neighborhood $|z| \sim \varepsilon^{1/2}$ this extremum is a minimum of the stochastic potential $U(X, z)$, as can be seen by differentiating twice and noting that expansion (10) around a stable stationary point implies that the coefficient a_1 in expansion (11) is always positive ($f' < 0$). Defining the shifted variable $Y = X - \bar{X}$, we obtain in this new variable

$$U(Y, z) = -\frac{(f'-\gamma)^2}{\gamma^2 g^2} f' \left(1 - \frac{2}{(2f'-\gamma)(f'-2\gamma)(f'-\gamma)} \times (-5f'^2 g' + 3f'f''g + 7\gamma f'g' - 2\gamma f''g - 2\gamma^2 g')z\right) Y^2 + \frac{1}{3}\frac{1}{\gamma^2 g^3}\frac{(f'-\gamma)^2}{(2f'-\gamma)(f'-2\gamma)} (20f'^3 g' - 28\gamma f'^2 g' - 14f'^2 f''g + 8\gamma^2 f'g' - 2\gamma f''g)\, Y^3 + \frac{z^2}{\gamma} + \frac{\varepsilon}{(2f'-\gamma)(f'-2\gamma)(f'-\gamma)} \times (-5f'^2 + 3f'f''g + 7\gamma f'g' - 2\gamma f''g - 2\gamma^2 g')\, z + \text{quartic terms} \tag{18}$$

The remarkable fact is that in these shifted variables the z^3 term cancels. We are now in a position to estimate integral (15). The dominant contribution to the integral comes from a neighborhood $|Y| \sim \varepsilon^{1/2}$, z being itself of order $\varepsilon^{1/2}$. In such a vicinity, the contribution of term Y^3 to the stochastic potential $U(z)$ is of order ε^2, so that it may be safely neglected. Hence, the integral (15) can be computed by elementary methods

$$P(z) = N\left(\frac{\pi\varepsilon}{f'}\right)^{1/2} \frac{\gamma g}{|f'-\gamma|}\left[1 - \frac{2}{(2f'-\gamma)(f'-2\gamma)(f'-\gamma)} \times (-5f'^2 g' + 3f'f''g + 7\gamma f'g' - 2\gamma f''g - 2\gamma^2 g')z\right]^{-1/2} \times \exp\left\{-\frac{1}{\varepsilon}\left[\frac{z^2}{\gamma} + \frac{\varepsilon}{(2f'-\gamma)(f'-2\gamma)(f'-\gamma)} \times (-5f'^2 g' + 3f'f''g + 7\gamma f'g' - 2\gamma f''g - 2\gamma^2 g')\, z + O(\varepsilon^2)\right]\right\} \tag{19}$$

By expanding the square root, we see that the term εz of $U(z)$ cancels and

$$U(z) = z^2/\gamma + O(\varepsilon^2) \tag{20}$$

in agreement with Eq. (16). Thus, our expansion procedure is consistent up to order $\varepsilon^{3/2}$.

Let us come now to the probability density for the x-process. We proceed in a way similar to the z-process: for given X, we calculate the extremum $\bar{z}$ of $U(X, z)$ given by Eqs. (12) and (13). Noting again that, since the coefficient a_3 in expansion (11) is always positive, this extremum is a minimum of the stochastic potential $U(X, z)$ in a neighborhood $|X| \sim \varepsilon^{1/2}$. Following the same procedure as before, we expand $U(X, z)$ around this local minimum $\bar{z}$ and neglect terms $(z-\bar{z})^3$ whose contribution would be of order ε^2. We finally obtain

$$\begin{aligned} U(X) = {} & \frac{(f'-\gamma)}{\gamma g^2} f'X^2 + \frac{1}{3\gamma g^3} \frac{(f'-\gamma)}{(2f'-\gamma)(f'-2\gamma)} \\ & \times (-12f'^3g' + 6f'^2f''g + 24\gamma f'^2g' \\ & - 9\gamma f'f''g - 2\gamma^2f'g' + 2\gamma^2f''g)\, X^3 \\ & + \frac{1}{g} \frac{\varepsilon}{(2f'-\gamma)(f'-2\gamma)} (4f'^2g' - 2f'f''g \\ & - 7\gamma f'g' + 2\gamma f''g + 2\gamma^2g')\, X + \text{quartic terms} \end{aligned} \tag{21}$$

As mentioned above, global normalization requires the presence of quartic terms in expression (21). Now, it is easy to see that up to first order in ε, these terms do not contribute to the shift of the extremum of $U(X)$. From Eq. (21) we deduce that the most probable value X_m is given by

$$\begin{aligned} X_m = {} & -\frac{1}{2} \frac{\varepsilon\gamma g}{(2f'-\gamma)(f'-2\gamma)(f'-\gamma)f'} \\ & \times (4f'^2g' - 2f'f''g - 7\gamma f'g' + 2\gamma f''g + 2\gamma^2g') \end{aligned} \tag{22}$$

This is a general expression valid for any value of the correlation time $1/\gamma$, provided that $\varepsilon\gamma^2$ is small. Let us comment on this expression, exploring first the vicinity of white noise. This limiting case corresponds to a vanishing correlation time. Letting $\gamma \to \infty$ in Eq. (22) yields

$$X_{mWN} = \frac{1}{2} \frac{gg'}{f'} \varepsilon \tag{23}$$

This result is in full agreement with the expression of the most probable value obtained from a one-dimensional Fokker–Planck equation in the Stratonovich version. We thus have a manifestation of the Wong–Zakai theorem, which states that if white noise is seen as the limit of colored noise with correlation time tending to zero, then the Stratonovich interpretation of the Fokker–Planck equation has to be adopted. We next consider the opposite limit of long correlation time. In this case, x becomes a fast relaxing variable as compared with $z(t)$. We can obtain the stationary probability density of the x-process by setting $\dot{x}=0$ in Eq. (1), expressing z in terms of x and using this relation to change variables in Eq. (16) (this is the well-known adiabatic elimination procedure[5] or the switching curve approximation[2]). From Eq. (1), we have

$$z = -\frac{1}{g^2}\left[f'gX + (\tfrac{1}{2}f''g - g'f')X^2\right] \tag{24}$$

Substituting this expression into Eq. (16) yields

$$P(X) = (\pi\varepsilon\gamma)^{-1/2}\frac{1}{g^2}\,|f'g' + (f''g - 2g'f')X|\,\exp\left(-\frac{1}{\varepsilon\gamma}\frac{f'^2}{g^2}X^2\right) \tag{25}$$

The maximum of this probability density is

$$X_{mAD} = -\frac{\gamma}{f'^3}(2f'g' - f''g)\varepsilon \tag{26}$$

It is immediately seen that up to first order in γ, the same expression is recovered by letting $\gamma \to 0$ in Eq. (22). We thus have the confirmation that our formula (22) gives accurate results for any value of γ. In the case of additive noise, it is easy to see that Eq. (22) taken in the limit of small correlation time yields the same result as Eq. (8.107) of Ref. 2 taken in the limit of small noise. It is worth emphasizing that due to the nonlinearities of the system, Eq. (22) predicts a shift in the most probable value even in the case of additive colored noise (in the limit of white noise this shift tends of course to zero). We have here evidence of a qualitative difference between additive white noise and colored noise. This result will be discussed further in Section 4.

3. A CLASS OF EXACTLY SOLVABLE MODELS

In this section, we consider a class of models for which it is possible to obtain the exact stationary solution to Eq. (5), and compare this exact result with the approximate results of the preceding section. The models we are dealing with satisfy a condition that reduces the problem to a linear one.[2,22] Starting from Eq. (1), it is easy to see that if functions $f(x)$ and $g(x)$ satisfy the condition

$$\frac{f(x)}{g(x)} = -\alpha \int^x \frac{1}{g(s)}\,ds + \beta; \qquad \alpha > 0 \tag{27}$$

the substitution

$$v = \int^x \frac{1}{g(s)}\,ds$$

yields a linear stochastic differential equation for the v-process. We thus have a two-dimensional Fokker–Planck equation, which can be solved exactly in the steady state. After straightforward algebra, we get the following expression for the stationary probability density of the x-process:

$$P(x) = N\,|R'(x)|\,\exp\left\{-\frac{\alpha}{\varepsilon}\frac{\alpha+\gamma}{\gamma}\left[R(x)-\frac{\beta}{\alpha}\right]^2\right\} \tag{28}$$

where

$$R(x) = \int^x \frac{1}{g(s)}\,ds$$

Because no approximation has been made, this probability density is globally normalizable (recall that $\alpha > 0$). The most probable values of $P(x)$ are solutions of

$$f(x_m) - \frac{\varepsilon}{2}\frac{\gamma}{\alpha+\gamma}\,g(x_m)\,g'(x_m) = 0 \tag{29}$$

If we introduce assumption (27) into Eq. (22), we recover Eq. (29), thus establishing the validity of our previous derivation. Let us treat a specific example, which will shed some further light on the general results of the preceding section.

The particular class of models described by

$$dx/dt = \lambda x - x^n + x^n z \tag{30}$$

meets condition (27) with $\alpha = \lambda(n-1)$, $\beta = 1$. It is worth noting that the substitution $x' = x^{(n-1)/n}$ transforms a model with a nonlinearity of degree n to one with nonlinearity of degree $n+1$. All the models of class (30) are thus equivalent, and it is sufficient to study a particular case to know the behavior of systems (30) for any value of n. In what follows, we treat the case $n=2$. The exact solution to Eqs. (8a) and (8b) is

$$U(x, z)=\frac{\lambda+\gamma}{\gamma^2}\left[(\lambda+\gamma)\left(\frac{1}{\lambda}-\frac{1}{x}\right)^2-2\lambda\left(\frac{1}{\lambda}-\frac{1}{x}\right)z+z^2\right]+2\varepsilon\ln x \quad (31)$$

while the exact probability density of the x-process is given by

$$P(x)=N\frac{1}{x^2}\exp\left[-\frac{\lambda(\lambda+\gamma)}{\varepsilon\gamma}\left(\frac{1}{\lambda}-\frac{1}{x}\right)^2\right] \quad (32)$$

To compare these expressions to the results of the previous section, we expand the stochastic potential and the probability density around the nontrivial stationary state $\bar{x}=\lambda$. This yields

$$\begin{aligned}U(X, z)=\frac{\lambda+\gamma}{\gamma^2}\bigg[&\frac{\lambda+\gamma}{\lambda^3}X^2-\frac{2}{\lambda}Xz+z^2-2\frac{\lambda+\gamma}{\lambda^4}X^3+\frac{2}{\lambda}X^2z\\&+\frac{3(\lambda+\gamma)}{\lambda^5}X^4-\frac{2}{\lambda^3}X^3z+\frac{2\varepsilon X}{\lambda(\lambda+\gamma)}-\frac{\varepsilon}{\lambda^2(\lambda+\gamma)}X^2\bigg]\end{aligned} \quad (33)$$

and

$$P(X)=N\exp\left[-\frac{1}{\varepsilon}\frac{\lambda+\gamma}{\lambda^3\gamma}\left(X^2-\frac{2}{\lambda}X^3+\frac{3}{\lambda^2}X^4+\frac{2\varepsilon\lambda^2\gamma^2}{\lambda+\gamma}X-\frac{\varepsilon\lambda\gamma^2}{\lambda+\gamma}X^2\right)\right] \quad (34)$$

One easily verifies that these expressions are identical to those given by Eqs. (12) and (13) up to order $\varepsilon^{3/2}$. In this particular case, it is possible to compute the expansion (33) from Eqs. (8a) and (8b) and check consistency up to second order in ε. Equation (34) also illustrates the fact that up to cubic terms, $P(x)$ is only locally normalizable, whereas the inclusion of quartic terms yields a globally normalizable probability density.

4. STOCHASTICALLY FORCED NORMAL FORM OF A PITCHFORK BIFURCATION

In this section, we apply the general results of Section 2 to study the stochastically perturbed pitchfork bifurcation:

$$dx/dt=\lambda x-x^3; \qquad \lambda>0 \quad (35)$$

As is well known Eq. (35) is the normal form of a wide class of dynamical systems undergoing such a transition. In the sequel, we study the effect of both additive and multiplicative colored noise on the system.

4.1. Multiplicative Noise

We consider

$$dx/dt = \lambda x - x^3 + xz \tag{36}$$

From Eq. (22) the most probable value turns out to be

$$x_m = \sqrt{\lambda} - \frac{1}{4}\frac{\varepsilon\gamma}{\sqrt{\lambda}}\frac{-4\lambda^2 + \lambda\gamma + \gamma^2}{(4\lambda+\gamma)(2\lambda+\gamma)(\lambda+\gamma)} \tag{37}$$

This problem has also been treated by Sancho *et al.*[10] on the basis of an expansion of the probability density in powers of the correlation time $\tau = 1/\gamma$ of the noise. Up to first order in τ and ε, formula (37) yields results analogous to those of Ref. 10. However, for larger values of τ, our results differ from those obtained in Ref. 10 under an *ad hoc* exponentiation assumption (i.e., the transformation of the first terms of a Taylor expansion into an exponential). From Eq. (37), we note that for any value of the control parameter λ, there always exists a value of the correlation time such that the shift of the most probable value vanishes. This has to be contrasted with the case of multiplicative white noise, where the shift always has a nonvanishing value. Here again we have a qualitative difference between white and colored noise.

4.2. Additive Noise

We have

$$dx/dt = dx - x^3 + z \tag{38}$$

The most probable value is now given by

$$x_m = \sqrt{\lambda} + \frac{3}{2}\frac{\gamma}{\sqrt{\lambda}}\frac{1}{(4\lambda+\gamma)(\lambda+\gamma)}\varepsilon \tag{39}$$

In contrast to the multiplicative case, the shift is always positive and vanishes in the limit of white noise. Therefore, additive colored noise tends to favor the nontrivial stationary state. In order to test this surprising result, we carried out a numerical simulation of Eq. (38) with $z(t)$ given by Eq. (2). Specifically, we create particular realizations of the x-process by standard Monte Carlo techniques.[10] For each realization, we integrate Eqs. (2) and (38) over 10,000 steps of size 0.005, obtaining in this way 11,100 independent realizations. $P_{st}(x)$ is represented by a histogram with a mesh size of 0.002. In order to ensure that $P_{st}(x)$ is indeed meaningful, we

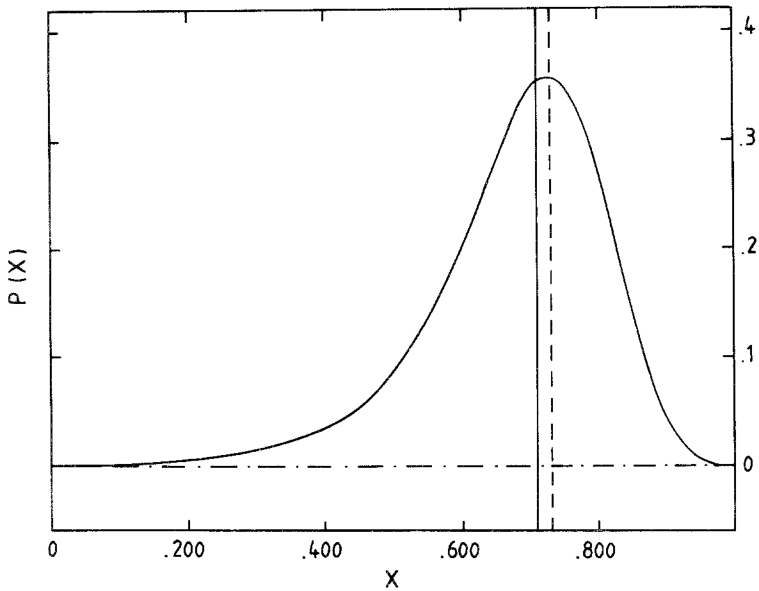

Fig. 1. Stationary probability density for $\varepsilon = 0.05$, $\lambda = 0.5$, $\gamma = 1$. (—) The deterministic steady state; (– –) the most probable value deduced from Eq. (22). The theoretical shift is 0.024, while numerical simulation gives 0.029.

chose the parameters ε, λ, and γ in such a way that the deterministic relaxation time is much smaller than the mean exit time from the attraction basin estimated from Kramer's theory. We consider the following situation: $\varepsilon = 0.05$, $\lambda = 0.5$, $\gamma = 1$ (for these values of the parameters, the mean first exit time is of the order of 240). The resulting histogram is plotted in Fig. 1. As expected from Eq. (39), Fig. 1 shows a shift in the most probable value. This shift is close to the theoretical prediction. Let us mention finally that the mean value $\langle x \rangle$ is shifted toward the trivial state (such a shift occurs already in the white noise case). The value of this shift is 0.030. From Eq. (21), it is easy to estimate this mean value by means of steepest descent methods. We obtain in this case a shift of the mean value of 0.026, in good agreement with the numerical simulation.

5. CONCLUSIONS

In this paper we have developed a canonical procedure for analyzing the effect of colored noise in nonlinear dynamical systems. In the case of one variable and a simple bifurcation to which our analysis has been limited so far, we have obtained model-independent expressions for the effect of noise on the stationary probability density. The extension of the analysis to higher codimension bifurcations or to multivariate systems giving rise, for instance, to Hopf bifurcation can be carried out along similar lines, the only difference being a greater algebraic complexity.

Another interesting feature of the method is the possibility of analyzing colored noise with correlation times of the order of unity. In this range qualitative differences with white noise have been brought out.

The extension of the present work to the time-dependent properties of the systems, in particular the corrections to the Kramer's formula of the mean exit time arising from colored noise, would be highly desirable.[24] It should be stressed, however, that to this end a global estimation of the stochastic potential would be necessary. An interesting problem that remains to be investigated is the process of ignition in explosive systems, which has been shown previously to be quite sensitive to stochastic disturbances.[25]

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